Grade-by-Grade Progression of Mathematical Instruction

Pre-Kindergarten

Students enter pre-kindergarten and find a well-planned, sequential math program awaiting, one that is embedded within hands-on, playful, interactive, largely concrete experiences. Students are encouraged to use their math words to communicate their observations.

The first step is to analyze, sort, classify, and count up to 5 with meaning. Students practice their numbers up-to-five fluency as they encounter and engage with circles, rectangles, squares, and triangles. With numbers to 5 understood, work can begin on extending "How Many" questions up to 10. The key here is to build from 5, using their fingers to support this perspective.

- 6 is 5 and 1.
- 7 is 5 and 2.
- 8 is 5 and 3, etc.

Thus, numbers 6-10 are 5 together with numbers 1-5, making the numbers to 10 familiar and manageable. Next, students measure length, weight, and capacity developing their word bank to include the language of comparison, "small, big, short, tall (length), heavy and light (weight), empty and full (capacity) while continuing to practice fluency with numbers to 10. With numbers 1-10 still developing, counting to 20 begins while addition and subtraction are initiated within classroom stories and playful contexts.

Kindergarten

The same themes as pre-kindergarten also run throughout kindergarten. It, too, starts out realistically with solidifying the meaning of numbers to 10 with a focus on graphing, relationships to 5 and growth and shrinking patterns to 10 of "1 more" and "1 less."

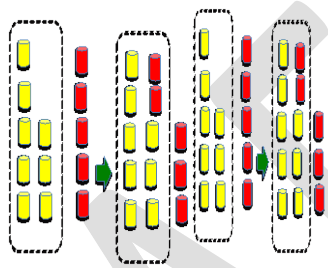
Next, students learn to identify and describe shapes while practicing their fluency with numbers to 10.

Two of the most crucial themes of students' math experience begin with measurement in kindergarten, the unit and comparison. Students use different units to measure length, weight and capacity, and explore the relationship of those units. Comparison begins with developing the meaning of the word "than": "taller *than*", "shorter *than*", "heavier *than*", "longer *than*", etc. With the word "than" concretely understood, the (at least!) 8-year curriculum teaching sequence for terms "more than" and "less than" can begin (a topic that culminates in middle school with "*y* is 2 less than 3 times as much as *x*."). The terms "more" and "less" are abstract later in kindergarten because they refer to numbers ("7 is 2 more than 5") rather than concrete measurements ("Jim is taller than John."). "1 more, 2 more, 3 more" lead into the addition fact fluencies (+1, +2, +3). Comparing numbers leads to looking at the numbers *that make up* a number ("3 is less than 7. 3 and 4 make 7. "). This, in turn, leads naturally to discussions of addition and subtraction.

With numbers 1-10 on firm ground, numbers 10-20 can be parsed as “10 together with a number from 1-10.” “12 is 2 more than 10.” Unlike the role of 5 in numbers 6-10, which loses significance as those numbers are shown in different configurations other than “5 and a number,” the number 10 is special; it *is* the anchor that will eventually become the “ten” unit in the place value system. The year rounds out by beginning explorations of concepts in area: that shapes can be composed of smaller shapes.

<u>Grade 1</u>

Work with “numbers to 10” continues to be a major stepping stone in learning the place value system. Unlike pre-kindergarten and kindergarten, this year starts out with exploring addition and subtraction within 10. Fluency with addition/subtraction facts, a major gateway to later grades, also begins right away with the intention of energetically practicing the entire year. The next major stepping stone is learning to group “10 ones” as a single unit: 1 ten.



$8 + 5 = 8 + (2 + 3) = (8 + 2) + 3 = 10 + 3 = 13$

Work begins slowly by “adding and subtracting across a 10”. Solutions like that shown above for 8 + 5 reinforce the need to “make 10.” This strategy of the “completion of a unit” empowers students in later grades to understand the “renaming” of the addition algorithm, to add 298 and 37 (i.e., 298 + 2 + 35), and add 4 ft. 8 in. and 5 in.

A module on expressing length measurement as numbers provides a few weeks in which to practice and internalize “making a 10” during students daily fluency activities. Introducing measurement early also has the added bonus of opening up the variety and types of word problems that can be asked throughout the year.

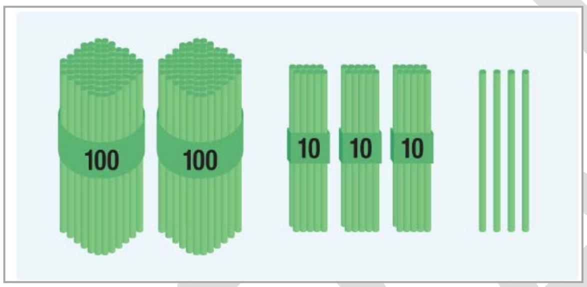
The focus of adding and subtracting within 40 is on establishing “1 ten” as a new unit. Before, students loosely grouped 10 objects to make 10. Now they transition to thinking of that 10 as a single unit (using 10 linker cubes stuck together, for example). Students begin to see in problems like 23+6 that they can mentally push the “2 tens” in 23 over to the side and concentrate on the familiar addition problem 3+6.

The focus of “adding and subtracting within 100” module is different than the “within 10” and “within 40” modules. Here the new level of complexity is to also introduce the addition and subtraction algorithms using simple examples and the familiar units of 10 made out of linker cubes.

Placed in between the two heavy-duty number modules is a module on geometry. The geometry module puts necessary internalization time between “within 40” and “within 100” modules and also gives students who may be more spatially oriented a chance to build confidence before heading back into arithmetic.

Grade 2

Students arrive in grade 2 having an extensive background working with numbers to 10. This year starts with establishing a motivating, differentiated fluency program in the first few weeks that will provide the amount of practice necessary for every student to reach mastery of the addition and subtraction facts to 20. Students next learn to measure using non-standard units (while continuing to practice fluency). Like the measurement module in grade 1, this module provides the necessary background to ask varied and multifaceted measurement problems throughout the year. The major underlying goal of the measurement module, however, is for students to learn the meaning of the word "unit," essentially by employing it repeatedly in describing length units, weight units, and capacity units. The idea of a unit is the most powerful concept in PK-5 mathematics—it is the unifying theme behind all explanations in arithmetic, measurement, and geometry in elementary school.



In particular, the unit plays a central role in the addition and subtraction algorithms of the next module. All arithmetic algorithms are manipulations of *place value units*: ones, tens, hundreds, etc. In grade 2, the place value units move from a proportional model (pictured above to the left) to a non proportional "number disk" model (pictured to the right), which has the versatility to be used through grade 5 for modeling very large numbers and decimals, allowing students greater facility with and understanding of mental math and the algorithms.

The work with units continues into the next module on multiplication as well. Making groups of 4 apples each establishes the unit "4 apples" (or just four) that can then be counted: 1 four, 2 fours, 3 fours, etc. Relating the new unit to the one used to create it develops the idea of multiplication: 3 groups of 4 apples equal 12 apples (or 3 fours is 12). The next module gives students another chance to practice their algorithms and problem solving skills with the most famous and most interesting units of all: dollars, dimes, and pennies.

The last module summarizes yearlong fluency work with telling time by pointing out two important relationships: an analog clock face is a "curved number line" (the precursor of a protractor) and that fractions naturally occur on a clock face (eg. half past the hour).

Grade 3

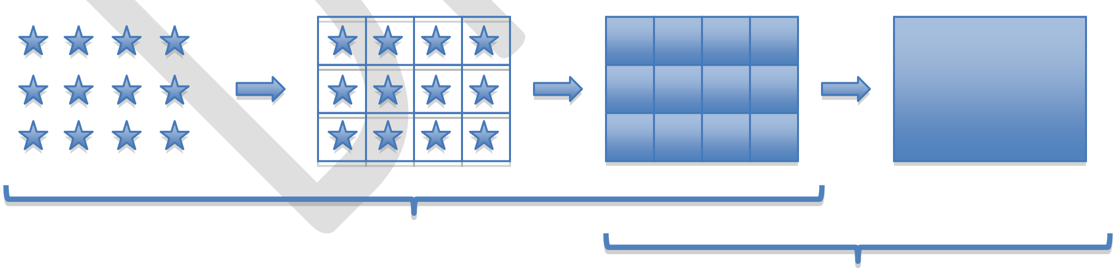
The first module of grade 3 continues and expands the idea of *units* realized in grade 2. Even rounding to the nearest hundred starts with the questions about place value units, "How many hundreds are in 1243? (12 hundreds) What is one more hundred? (13 hundreds) What is halfway between 1200 and 1300? (1250)" One of the goals of this module is for students to think of a number in numerous ways using different place value units, i.e., 2843 can be thought of as

> 2 thousand 843 ones
> 28 hundreds 43 ones
> 285 tens 3 ones
> 28 hundreds 4 tens 3 ones
> etc.

This flexibility is preparation for further development of the addition and subtraction algorithms, but it also highlights multiplicative nature of the place value system: Counting by tens is multiplying by 10: 4 tens = 4 × 10 = 40.

The next module builds upon the same multiplicative thinking with units to begin learning the multiplication and division facts, but only for factors of 2, 3, 4, 5, and 10. The restricted set of facts makes learning manageable for beginners while providing enough examples to start word and measurement problems involving weight, capacity, and time in the third module. The measurement module again plays the role of providing students with "internalization time" before students start into the remaining facts in the fourth module.

The "2,3,4,5, and 10 facts" and the "6, 7, 8, 9 facts" modules both play key roles in preparing students to learn about area. Students often find it difficult to distinguish the different squares in a rectangular array area model, count them, and recognize that the count is related to multiplication until they have worked extensively with a Rekenrek and/or pictures of rectangular arrays involving objects only (stars, disks, etc.).



Module 2 and Module 4

Module 5: Multiplication and Area

Both modules provide important, sustained, work with both concrete and pictorial models to prepare students for area in the fifth module.

Area is the number of non-overlapping *area units* in a given shape. When that shape is a rectangle with whole number side lengths, it is easy to partition the rectangle into squares with equal areas (like the 3rd rectangle above). The area of each square is then a fraction of the area of the rectangle, which links the sixth module to the fifth module and to the last module of grade 2 (as well as the last module in kindergarten). The goal of the fifth module, of course, is for students to transition from thinking of fractions as parts of a figure to points on a number line. To make that jump students once again have to think of fractions as special types of units: Forming fractional units is exactly the same as what was done for multiplication, but the "group" is now allowed to be the amount when a whole unit is subdivided equally: "1 fourth" is the length of a segment on the number line such that the length of 4 concatenated fourth segments on the line equals 1. Once the unit "1 fourth" has been established, counting them is as easy as counting whole numbers: 1 fourth, 2 fourths, 3 fourths, 4 fourths, 5 fourths, etc. By the end of the year, students have had enough work with both linear and area measurement models to begin to study the (non) relationship between the perimeter and area of a figure, the major topic in the last module of grade 3.

Grade 4

In today's world, "big units" are quite common in our daily lives. For example, movies take about a gigabyte (1,000,000,000 bytes) to store on a computer while songs take about 1 megabyte (1,000,000 bytes). To understand these big numbers, the students rely upon previous mastery of rounding and the addition and subtraction algorithms. In a sense the algorithms have come full circle: In grades 2 and 3 the algorithms were the *abstract* idea students were trying to learn, but by grade 4 the algorithms have become the *concrete* knowledge students are relying upon to understand new ideas. The algorithms continue to play a part in the next module on unit conversions. This module is intentionally designed to be repetitive to help students draw similarities between:

10 ones = 1 ten

100 ones = 1 hundred
100 cm = 1 m

1000 ones = 1 thousand
1000 m = 1 km
1000 g = 1 kg
1000 mL = 1 L.

Measurement problems again act as the "glue" that binds knowledge of the algorithms, mental math, place value, and real-world applications together into a coherent whole.

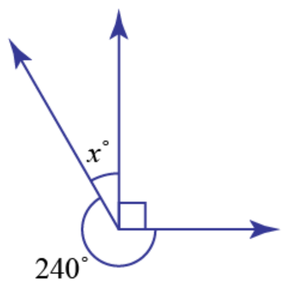
In the next module, compound measurement units help provide the concrete foundation behind the distributive property in the multiplication algorithm: 4 × (1 m 2 cm) can be made physical using ribbon where it is easy to see the 4 copies of 1 m and the 4 copies of 2 cm. Likewise, 4 × (1 ten 2 ones) = 4 tens 8 ones.

There are two pathways students travel in this curriculum to prepare for algebra in middle school. Both are extremely important. The first is solving word problems using bar diagrams (pictorial algebra). The second is solving unknown angle problems, which are first introduced in the fourth module. Students learn how to measure angles in degrees using a protractor. They also learn basic facts about angles:

1. vertical angles are equal,
2. the sum of angle measurements on a line is 180 degrees, and
3. the sum of angle measurements around a point is 360 degrees.

Armed with just these three facts (and the obvious one that angle measures of adjoining angles add), students are able to generate and solve equations that make sense:

Find the unknown angle x.



$$x + 240 + 90 = 360,$$
$$x + 330 = 360,$$
$$x = 30.$$

Geometry is the key that unlocks algebra for students *because it is visual.* The *x* clearly stands for a specific number: If a student wanted to, he or she could place a protractor down on that angle and measure it to find *x*. But doing so destroys the joy of solving the puzzle and deducing the answer for themselves.

We use fractions when there is a given unit, the *whole unit*, but we want to measure using a smaller unit, called the *fractional unit*. Students have been carefully exposed to many small units up to this point in the year:

360 degrees in 1 complete turn,
1000 g in 1 kilogram,
1000 mL in 1 liter, etc.

The beauty of fractional units is that, once defined, they behave just like whole number units:

- "3 fourths + 5 fourths = 8 fourths" like "3 apples + 5 apples = 8 apples," and
- "3 fourths × 4 = 12 fourths" like "3 apples × 4 = 12 apples."

Decimals start with the realization that decimal place value units are just special fractional units: 1 tenth = 1/10, 1 hundredth = 1/100, etc. Fluency plays an important role in both of these topics as students learn to relate 3/10 = 0.3 = 3 tenths.

The year ends with an exploratory module on multiplication. Students have been practicing the algorithm for multiplying by a 1-digit number since the third module. The goal here is to structure opportunities for them to "discover" ways to multiply 2-digit × 2-digit numbers by using their tools (place value tables, area models, bar models, number disks, the distributive property and equations, etc.).

Grade 5

Students' experiences with the algorithms as ways to manipulate place value units in grades 2-4 really begins to pay dividends in grade 5. Whole number patterns with number disks on the place value table are easily generalized to decimal numbers. As students work word problems with measurements in the metric system, where the same patterns occur, they begin to appreciate the value and the meaning of decimals. Fractions of the form 1/10, 1/100, 1/1000 also play a prominent role in the first module and are used in investigating patterns on the place value table.

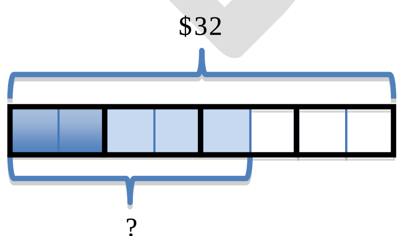
The first module provides students a chance to practice and hone their skills at multiplying and dividing (decimal) numbers by 1-digit whole numbers. They are now ready to generalize the 1-digit algorithms to the multi-digit versions. For multiplication, students must grapple with and fully understand the distributive property (one of the key reasons for *teaching* the multi-digit algorithm). While the multi-digit multiplication algorithm is a straightforward generalization of the 1-digit multiplication algorithm, the division algorithm with 2-digit divisor requires far more care to teach because students have to also learn estimation strategies, error correction strategies, and the idea of successive approximation (all of which are central concepts in math, science, and engineering).

The work with place value units in the first two modules paves the path to fraction units and arithmetic with fractions, for example,

"8 ninths ÷ 2 = 4 ninths" is the same as "8 tenths ÷ 2 = 4 tenths."

The new level of complexity in the third and fourth module is relating different fractional units to a common fractional unit: 1 third + 1 fourth = 4 twelfths + 3 twelfths = 7 twelfths. Relating different fractional units together back to the whole unit requires extensive work with area and number line models, fluency, and bar diagrams used in word problems. Bar diagrams start in this curriculum in the early grades and grow in power and usefulness as students progress through the grades. At the heart of a bar diagram is again the idea of *forming units*. In fact, forming units to solve word problems is one of the most powerful examples of the unit theme and are particularly helpful for understanding fraction arithmetic, as in the following example:

Jill had $32. She gave – of her money to charity and – of her money to her brother. How much did she give altogether?



Jill gave $20 altogether.

Solution with units:

8 units = $32
1 unit = $4
5 units = $20.

with arithmetic:

– – – – –

– .

The fraction module prepares students through the daily use of area models for an in depth discussion of area in the next module. But the module on area and volume also reinforces the work done in the fraction module: questions can now be asked about how the area changes when a rectangle is scaled by whole or fractional scale factor. Measuring volume once again highlights the unit theme as a unit cube is chosen to represent a volume unit and used to measure the volume of simple shapes made out of rectangular prisms.

Scaling is returned to in the last module on coordinate plane. Ever since the growth and shrinking patterns where first introduced in kindergarten, students have been using bar graphs to display data and patterns. All that work with bar graphs over the years has set the stage for line plots, which is both the natural extension of bar graphs and the precursor to linear functions. It is in this final module of K-5 that a simple line plot of a straight line is presented on a coordinate plane and students are asked about how the scaling relationship between the increase in the units of the vertical axis for 1 unit of increase in the horizontal axis—the first hint of slope and the beginning of the "Story of Ratios" in middle school.